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The Left Shift Semigroup Approach to Stability of Distributed Systems*

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DEDICATED TO MY TEACHER PROFESSOR D. G. LAMPARD
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1. INTRODUCTION

The aim of this paper is to study weak and strong stabilities of linear distributed systems on Hilbert space.

It is well known that the necessary and sufficient conditions of Lyapunov for a matrix to be stable can be extended to exponential stability of semigroups of bounded linear operators, while no such conditions exist for weak and strong stabilities. Our study is motivated by the fact that exponential stability is hard to come by for many distributed systems. This is also the case for exponential stabilizability.

The main theme of our study is to establish relationships between a stable C_0 (strongly continuous) semigroup of bounded linear Hilbert space operators and the Left Shift semigroup over the space $L^2([0, \infty), K)$, where K is an auxiliary Hilbert space. The left shift semigroup is known to be strongly stable while its adjoint is an isometric semigroup which is, at the same time, weakly stable.

We begin in Section 2 by showing that every exponentially stable semigroup is a quasi-affine transform of a contraction semigroup. Therefore, if the exponentially stable semigroup also defines an equivalent norm then it is similar to a contraction semigroup. Moreover, it can be isometrically represented as a "part", i.e., the restriction to an invariant subspace, of the left shift semigroup. A necessary and sufficient condition for an exponen-

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tially stable semigroup to define an equivalent norm was given by Pazy. It turns out that this condition is an exact controllability condition.

Next we turn to strong stability of the class of uniformly bounded semigroups. A necessary and sufficient condition is found. This, in the case of a contraction semigroup, leads directly to a relationship between a strongly stable contraction semigroup and the left shift semigroup. However, if the contraction semigroup is not strongly stable, our condition results in a sufficient condition for weak stability.

Finally, we concentrate on weak and strong stabilities of the class of uniformly bounded and non-contractive semigroups which is, at the same time, a quasi-affine transform of a contraction semigroup. A general sufficient condition for weak stability is found. This requires, in particular, approximate controllability and "Weak-L^2-Stability" for a "fixed" element. This last qualification is imposed since, in general, "Weak-L^p-Stability" is equivalent to exponential stability.

We close the paper by studying the equation

$$[PAx, x] + [x, PAx] = -\|x\|^2_{\text{new}}, \qquad \text{for } x \text{ in } D(A), \text{ the domain of } A,$$

where A generates a uniformly bounded semigroup over a Hilbert space H equipped with the norm $\|\cdot\|$, while $\|\cdot\|_{\text{new}}$ is a new norm which is not equivalent to the original norm. Sufficient conditions for weak and strong stabilities are found, depending on whether the new norm is defined on all of H, or just on the domain of A. These results, we claim, are Lyapunov-type results for weak and strong stabilities.

2. The Main Results

In the following we deal with strongly continuous semigroups, i.e., of the class C_0, of bounded linear operators over a Hilbert space. Inner product and norm are denoted by $[\cdot, \cdot]$ and by $\|\cdot\|$, respectively. A semigroup $T(t)$, $t \geq 0$, over a Hilbert space is simply denoted by $[T(t)]$.

We begin by recalling the following key notions of stability.

Definition. A semigroup $[T(t)]$ over H is called: (i) *e(exponentially)-stable* if there exist contants $M \geq 1$ and $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$, $t \geq 0$; (ii) *s(strongly)-stable* if, for x in H: $\|T(t)x\| \to 0$, $t \to \infty$; and (iii) *w(weakly)-stable* if, for x and y in H: $[T(t)x, y] \to 0$, $t \to \infty$.

It is evident that the three types of stability are equivalent as soon as the dimension of H is finite.

The following results on e-stability are due to Datko [1].

THEOREM 1. *For a C_0-semigroup $[T(t)]$, with generator A, over a Hilbert space H, the following conditions are equivalent*:

(i) $[T(t)]$ *is e-stable*;

(ii) *There exists a positive operator P on H such that*

$$2\,Re\cdot[PAx, x] = -\|x\|^2, \qquad \text{for } x \text{ in } D(A), \text{ the domain of } A; \quad (2.1)$$

(iii) *For x in H*: $\int_0^\infty \|T(t)x\|^2\,dt < \infty$.

Note that condition (ii) is an analog of the Lyapunov equation for a matrix A, i.e., $PA + A^*P = -I$, while condition (iii) simply implies that $T(t)x$ is an element of the space $L^2([0, \infty), H)$.

Now let K be a Hilbert space; the Left Shift semigroup over the space $L^2([0, \infty), K)$ is denoted by $[L(t)]$ and is defined by

$$L(t)f = g, \qquad g(\tau) = f(\tau + t), \qquad \text{for} \quad t, \tau \geqslant 0.$$

It is plain that this is a semigroup of contractions, $\|L(t)\| \leqslant 1$, $t \geqslant 0$. Moreover, it is also strongly stable. The adjoint semigroup $[L(t)^*]$ is called the Right Shift, and it is a semigroup of isometries. Hence the right shift semigroup is *not* strongly stable, but it is weakly stable.

We now show that there is a class of exponentially stable semigroups which is closely "related" to the Left Shift semigroup.

Let $[T(t)]$ be an exponentially stable semigroup over H then, by (2.1), for x in $D(A)$:

$$2\,Re\cdot[PAT(t)x, T(t)x] = \frac{d}{dt}[PT(t)x, T(t)x] = -\|T(t)x\|^2, \qquad t \geqslant 0.$$

Therefore, for x in H,

$$[PT(t)x, T(t)x] - [Px, x] = -\int_0^t \|T(\tau)x\|^2\,d\tau, \qquad t \geqslant 0,$$

by the fact that an e-stable semigroup is uniformly bounded, $\|T(t)\| \leqslant M$, and the domain $D(A)$ is dense in H. It then follows that

$$[PT(t)x, T(t)x] \leqslant [Px, x], \qquad \text{for} \quad x \text{ in } H, \quad (2.2)$$

or

$$\|QT(t)x\| \leqslant \|Qx\|, \quad (2.3)$$

where $Q^2 = P$. For $t \geqslant 0$ define $C(t)$ by

$$C(t)\,Qx = QT(t)x. \quad (2.4)$$

Then, for x in H, and for $t, \tau \geqslant 0$,

$$C(\tau)\, C(t)\, Qx = C(\tau)\, QT(t)x = QT(\tau)\, T(t)x = QT(\tau + t)x = C(\tau + t)\, Qx.$$

This together with (2.3) shows that $C(t)$, $t \geqslant 0$, is a semigroup of contractions on the range of Q which is dense, since P is positive. Hence $C(t)$ is well-defined on all of H. The semigroup $[T(t)]$ is therefore called a *quasi-affine transform* of a contraction semigroup, [2].

Now, it follows from Datko's Theorem that the positive operator P is given by

$$[Px, x] = \int_0^\infty \|T(t)x\|^2\, dt, \qquad \text{for } x \text{ in } H. \tag{2.5}$$

Define, for x and y in H,

$$[x, y]_P = [Px, y]. \tag{2.6}$$

Then (2.5) becomes

$$\|x\|_P^2 = \int_0^\infty \|T(t)x\|^2\, dt, \qquad \text{for } x \text{ in } H. \tag{2.7}$$

Suppose now that the exponentially stable semigroup $[T(t)]$ is such that the norm $\|\cdot\|_P$ induced by $[\cdot, \cdot]_P$ is *equivalent* to the original norm $\|\cdot\|$. Then the positive operator P is invertible. Therefore it follows from (2.4) that $[T(t)]$ is *similar* to a contraction semigroup. However, more can be concluded from (2.7).

First, let $V\colon H \to L^2([0, \infty), H)$ be defined by

$$(Vx)(t) = T(t)x, \qquad \text{for} \quad t \geqslant 0 \qquad \text{and for } x \text{ in } H. \tag{2.8}$$

Then V is a bounded linear operator on H. Moreover,

$$\|Vx\|_{L^2}^2 = \int_0^\infty \|T(t)x\|^2\, dt = \|x\|_P^2, \qquad \text{for } x \text{ in } H. \tag{2.9}$$

This shows that V is an isometry on H equipped with the equivalent norm $\|\cdot\|_P$. We have for x in H and for $t, \tau \geqslant 0$,

$$[VT(\tau)x](t) = T(t)\, T(\tau)x = T(t + \tau)x = [L(\tau)\, Vx](t),$$

where, as before, $[L(\tau)]$ denotes the Left Shift semigroup on $L^2([0, \infty), H)$. Thus we have shown that

$$VT(\tau)x = L(\tau)\, Vx, \qquad \text{for } x \text{ in } H \qquad \text{and for} \quad \tau \geqslant 0. \tag{2.10}$$

This shows that the closed subspace VH is invariant under the Left Shift semigroup and, since V is an isometry,

$$T(\tau) = V^*L(\tau)V, \qquad \tau \geqslant 0.$$

We summarize the above in the next Theorem.

THEOREM 2. *Let $[T(t)]$ be an exponentially stable semigroup over a Hilbert space H. Then $[T(t)]$ is a quasi-affine transform of a contraction semigroup. In particular, if $[T(t)]$ defines an equivalent norm on H then it is similar to a contraction semigroup. Moreover, in this case, it can be isometrically represented by the restriction of the Left Shift semigroup over $L^2([0, \infty), H)$ to an invariant subspace.*

Necessary and sufficient conditions for an e-stable semigroup to define an equivalent norm were given by Pazy [3]. Combining Pazy's results with Theorem 2 we obtain.

COROLLARY 1. *Let $[T(t)]$ be an e-stable semigroup over H. If for x in H and for some $t_0 > 0$: $\|T(t_0)x\| \geqslant c\,\|x\|$ for some $c > 0$, then $[T(t)]$ is similar to a contraction semigroup. This is also the case if, in particular, $T(t)H$ for $t > 0$ is dense in H, and the semigroup can be extended to a group of bounded linear operators over H.*

Note that an exponentially stable semigroup is uniformly bounded. Thus the above theorem identifies a class of uniformly bounded semigroups which is similar to a contraction semigroup. The question whether or not every uniformly bounded "discrete" semigroup T^n, $n \geqslant 0$, is similar to a contraction was first posed by Sz-Nagy [4]. A counter example to this was given by Foguel [5]. The counter example to the C_0-semigroup analog of Sz-Nagy's question was given by Packel [6]. It is of no surprise to see that e-stability can be related to similarity to a contraction, since we have seen that an e-stable semigroup is already a quasi-affine transform of a contraction semigroup. What is interesting here is the isometric equivalence to a "part" of a Left Shift semigroup which, as we see, is also the case for a strongly stable contraction semigroup.

Another way of seeing the relationship between e-stability and similarity to a contraction semigroup is as follows. Let $[T(t)]$ be uniformly bounded and suppose that it is similar to a contraction semigroup $[C(t)]$, say,

$$T(t) = S^{-1}C(t)S, \qquad \text{for} \quad t \geqslant 0.$$

Then, clearly,

$$Re \cdot [SAS^{-1}x, x] \leqslant 0, \qquad \text{for } x \text{ in } D(A),$$

or

$$Re \cdot [S^*SAz, z] \leqslant 0, \qquad \text{for } z \text{ in } D(A), \tag{2.11}$$

where the self-adjoint operator S^*S is strictly positive: $[S^*Sx, x] \geqslant k \|x\|^2$ for some $k > 0$ and for all x, by the fact that S is invertible. Conversely, if (2.11) holds for some strictly positive operator S^*S then $[T(t)]$ is similar to a contraction semigroup. We conclude that

PROPOSITION 1. *A uniformly bounded semigroup $[T(t)]$, with generator A, on H is similar to a contraction semigroup on H, if and only if there is a strictly positive operator P such that, for x in $D(A)$:*

$$2\, Re \cdot [PAx, x] \leqslant 0. \tag{2.12}$$

It is evident that if $[T(t)]$ is e-stable then P exists; in addition, if Pazy's conditions are satisfied then P is strictly positive. Hence the exponentially stable semigroup $[T(t)]$ is similar to a contraction semigroup. We note that similarity to a contraction semigroup need not imply e-stability.

To proceed further, we recall the following results on exact controllability, [7].

THEOREM 3. *The system $(A, B): \dot{x} = Ax + Bu$, where B is linear bounded from a Hilbert space U to H and is exact controllable if and only if there exist $t_0 > 0$, and $\alpha > 0$ such that*

$$\int_0^{t_0} \|B^*T(t)^* x\|^2 \, dt \geqslant \alpha \|x\|^2, \qquad \text{for } x \text{ in } H, \tag{2.13}$$

where $[T(t)^*]$ is the adjoint of the semigroup generated by the operator A.

Now let $[T(t)]$, with generator A, be an exponentially stable semigroup and suppose that the pair (A^*, I) is exact controllable. Then by the above Theorem, for x in H:

$$\int_0^{t_0} \|T(t)x\|^2 \, dt \geqslant \alpha \|x\|^2, \qquad \text{for some} \quad t_0 > 0 \qquad \text{and some} \quad \alpha > 0.$$

Therefore, since $[T(t)]$ is e-stable, for x in H:

$$\int_0^{\infty} \|T(t)x\|^2 \, dt \geqslant \int_0^{t_0} \|T(t)x\|^2 \, dt \geqslant \alpha \|x\|^2.$$

This shows that the operator P defined by (2.5) is strictly positive; hence it defines an equivalent norm.

Conversely, if P defines an equivalent norm, then by Pazy's results, there are $t_0 > 0$ and $c > 0$ such that

$$c\,\|x\| \leqslant \|T(t_0)x\|, \qquad \text{for } x \text{ in } H. \tag{2.14}$$

Therefore, for $t < t_0$,

$$c\,\|x\| \leqslant \|T(t_0 - t)\,T(t)x\| \leqslant M\,\|T(t)x\|,$$

by the fact that $[T(t)]$ is uniformly bounded. It then follows that the pair (A^*, I) is exact controllable, by Theorem 3.

We have therefore shown that

PROPOSITION 2. *Let $[T(t)]$ be an exponentially stable semigroup over H. Then the integral $\int_0^\infty \|T(t)x\|^2\,dt$ defines an equivalent norm if and only if the pair (A^*, I) is exact controllable. Consequently, a sufficient condition for an e-stable semigroup $[T(t)]$, with generator A, to be similar to a contraction semigroup is that the pair (A^*, I) be exact controllable.*

It is of interest to observe from the above that condition (2.14) of Pazy is an exact controllability condition. Indeed, it is equivalent to exact controllability of the pair (A^*, I).

We now turn to strong stability and its relationships to the Left Shift semigroup. We have, for x in the domain of A,

$$\frac{d}{dt}\|T(t)x\|^2 = 2\,Re \cdot [AT(t)x, T(t)x].$$

Therefore, for $t \geqslant 0$ and for x in $D(A)$,

$$\|T(t)x\|^2 - \|x\|^2 = \int_0^t 2\,Re \cdot [AT(\tau)x, T(\tau)x]\,d\tau. \tag{2.15}$$

Thus, if the semigroup is strongly stable then

$$\|x\|^2 = -\int_0^\infty 2\,Re \cdot [AT(t)x, T(t)x]\,dt, \qquad \text{for } x \text{ in } D(A).$$

Conversely, if this holds and if the semigroup is uniformly bounded then it follows from (2.15) that, for x in $D(A)$: $\lim_{t\to\infty} \cdot \|T(t)x\| = 0$. Hence $[T(t)]$ is s-stable, by the fact that the domain of A is dense in H. We have

THEOREM 4. *A uniformly bounded semigroup $[T(t)]$ with generator A is stronly stable if and only if, for x in the domain of A,*

$$\|x\|^2 = -\int_0^\infty 2\,Re \cdot [AT(t)x, T(t)x]\,dt. \tag{2.16}$$

It is plain that if the integral on the right-hand side of (2.16) defines the square of the norm of some space $L^2([0, \infty), K)$ say, then we can, as in the case of (2.7), "connect" $[T(t)]$ to the Left Shift semigroup. This is certainly the case if $[T(t)]$ is a semigroup of contractions. For then its generator A is dissipative. Hence we can define a new norm $\|\cdot\|_n$ for $D(A)$:

$$\|x\|_n^2 = -([Ax, x] + [x, Ax]) \geqslant 0.$$

Therefore (2.16) becomes

$$\|x\|^2 = \int_0^\infty \|T(t)x\|_n^2 \, dt, \qquad \text{for } x \text{ in } D(A). \tag{2.17}$$

As before, let V be the operator from $D(A)$ into $L^2([0, \infty), K)$—here K is the completion of $D(A)$ in the new norm, modulo the null elements—defined by

$$(Vx)(t) = T(t)x.$$

Then it is plain from (2.17) that V is an isometry on $D(A)$, hence on all of H, by continuity and by the fact that $D(A)$ is dense in H. Let $[L(t)]$ again denote the Left Shift semigroup over $L^2([0, \infty), K)$; then, as in the above, it is easy to see that

$$VT(t)x = L(t)\, Vx, \qquad \text{for } x \text{ in } H.$$

This shows that the closed subspace VH is invariant for the Left Shift semigroup.

We have therefore shown that

THEOREM 5. *A contraction semigroup is strongly stable if and only if it can be isometrically represented as the restriction of the Left Shift semigroup, on $L^2([0, \infty), K)$, where K is an auxiliary Hilbert space, to an invariant subspace.*

Actually more can be obtained for the case of a contraction semigroup. Returning to (2.15) and suppose that $[T(t)]$ is contractive. Then letting t go to ∞ we obtain

$$\|x\|^2 - \lim_{t \to \infty} \|T(t)x\|^2 = \int_0^\infty \|T(t)x\|_n^2 \, dt, \qquad \text{for } x \text{ in } D(A). \tag{2.18}$$

Now, since the function $\|T(t)x\|$ is non-increasing, we can define a non-negative contraction C by

$$\lim_{t \to \infty} T(t)^* \, T(t) = C^2.$$

Therefore (2.18) becomes

$$[Px, x] = \int_0^{\infty} \|T(t)x\|_n^2\, dt, \qquad \text{for } x \text{ in } D(A), \tag{2.19}$$

where

$$P = I - C^2$$

is non-negative since C is a contraction. It is clear that (2.19) is analogous to (2.5). Moreover, it follows that, for x in the domain of A,

$$[PT(t)x, T(t)x] = \int_t^{\infty} \|T(\tau)x\|_n^2\, d\tau, \qquad t \geqslant 0. \tag{2.20}$$

Therefore

$$\text{for } x \text{ in } D(A)\colon \lim_{t \to \infty} [PT(t)x, T(t)x] = 0.$$

Thus, if P is positive then $[T(t)]$ is weakly stable, by the fact that the domain of A is dense and the semigroup is uniformly bounded. Moreover, by differentiating (2.20) and setting $t = 0$ we obtain the following "Lyapunov-type" equation:

$$2\, Re \cdot [PAx, x] = -\|x\|_n^2, \qquad \text{for } x \text{ in } D(A).$$

We summarize the above in the next theorem.

THEOREM 6. *Let $[T(t)]$ be a contraction semigroup over H and let A denote its generator. Then there exists a unique non-negative operator P such that*

$$2\, Re \cdot [PAx, x] = -\|x\|_n^2, \qquad \textit{for } x \textit{ in } D(A). \tag{2.21}$$

Moreover, if P is positive then the semigroup is weakly stable.

We note that if P is positive then the contraction semigroup $[T(t)]$ is also completely nonunitary; i.e., the trivial subspace is the only reducing subspace on which the semigroup is unitary. For more details we refer to [8].

Now it is plain from (2.21) that, for a contraction semigroup $[T(t)]$, there always exists a non-negative operator P such that

$$[PT(t)x, T(t)x] \leqslant [Px, x], \qquad \text{for } x \text{ in } H \qquad \text{and for} \quad t \geqslant 0. \tag{2.22}$$

This is also the case for an e-stable semigroup, except in the case where the operator P is positive. These observations suggest that we should investigate the class of C_0-semigroups for which there exists $P>0$ so that (2.22) holds. This class is, as we have seen, the class of quasi-affine transforms of contraction semigroups.

LEMMA 1. *Let* $[T(t)]$ *be a uniformly bounded semigroup over* H. *Suppose that* (2.22) *holds for some* $P>0$. *Then* $[T(t)]$ *is weakly stable if and only if*

$$[PT(t)x, y] \to 0, \qquad t \to \infty, \qquad \text{for } x \text{ and } y \text{ in } H. \tag{2.23}$$

Therefore, if

$$[PT(t)x, T(t)x] \to 0, \qquad t \to \infty, \qquad \text{for } x \text{ in } H, \tag{2.24}$$

then $[T(t)]$ *is weakly stable.*

Proof. The proof is all but trivial. Weak stability clearly implies (2.23). If (2.23) is true then

$$[T(t)x, z] \to 0, \qquad t \to \infty, \qquad \text{for } x \text{ in } H \text{ and } z \text{ in the range of } P.$$

Then, since the range of P is dense and $[T(t)]$ is uniformly bounded, the semigroup is weakly stable, as expected. Finally, (2.24) implies (2.23) and the proof is completed.

Note that if (2.24) holds and if P is strictly positive then $[T(t)]$ is strongly stable. However, we are not interested in this case since the semigroup is now similar to a contraction semigroup.

We now state a weakly stable result.

THEOREM 7. *Let* $[T(t)]$ *be a uniformly bounded semigroup over* H. *If for some* $P>0$: $[PT(t)x, T(t)x] \leqslant [Px, x]$, *for* x *in* H *and for* $t \geqslant 0$, *and if*

$$[Px, x] = \int_0^\infty -2\, Re \cdot [PAT(t)x, T(t)x]\, dt, \qquad \text{for } x \text{ in } D(A), \tag{2.25}$$

then $[T(t)]$ *is weakly stable.*

Proof. As in the proof of (2.16), (2.25) is necessary and sufficient for

$$[PT(t)x, T(t)x] \to 0, \qquad t \to \infty, \qquad \text{for } x \text{ in } H.$$

Hence, by Lemma 1, $[T(t)]$ is weakly stable.

We have, as a Corollary of Theorem 7

COROLLARY 2. *Let $[T(t)]$ be a uniformly bounded semigroup with generator A over H. Let b be a fixed element of H and suppose that, for x in H:*

$$[T(t)x, b] = 0 \qquad \textit{for} \quad t \geqslant 0 \Rightarrow x = 0, \tag{2.26}$$

and

$$\int_0^\infty |[T(t)x, b]|^2 \, dt < \infty. \tag{2.27}$$

Then $[T(t)]$ is weakly stable.

Proof. Let P_b be defined by

$$[P_b x, x] = \int_0^\infty |[T(t)x, b]|^2 \, dt, \qquad \text{for } x \text{ in } H.$$

Then, clearly, P_b is a non-negative operator on H; moreover it is also positive, by (2.26). Next, we have, for x in the domain of A,

$$2\,Re \cdot [P_b A T(t)x, T(t)x] = -|[T(t)x, b]|^2, \qquad t \geqslant 0. \tag{2.28}$$

Hence $[T(t)]$ is weakly stable by Theorem 7. This finishes the proof of the Corollary.

Note that $[T(t)]$ is w-stable if and only if $[T(t)^*]$ is. Moreover, (2.26) simply implies that the pair (A^*, b) is approximately controllable. Hence

COROLLARY 3. *Let $[T(t)]$ be uniformly bounded with generator A on H. If for a fixed element b of H:* (i) *the pair (A, b) is approximately controllable, and* (ii) *for each x in H:*

$$\int_0^\infty |[T(t)^* x, b]|^2 \, dt < \infty. \tag{2.29}$$

Then $[T(t)]$ is weakly stable.

Note that the operator P_b satisfies the "Lyapunov-type" equation

$$2\,Re \cdot [P_b A x, x] = -|[x, b]|^2 \qquad \text{for } x \text{ in } D(A). \tag{2.30}$$

In fact, (2.27) is necessary and sufficient for the existence of a non-negative solution of (2.30). We refer to [8] for more details on Lyapunov-type equations. Note also that if (2.27) holds for every b in H then the semigroup $[T(t)]$ is called "Weak-L^2-Stable". Recently Weiss [9] has shown that Weak-L^p-Stability is equivalent to exponential stability. This conjecture was first posed by Pritchard and Zabczyk [10].

To proceed further we define, for x and y in H,

$$[x, y]_P = [Px, y]$$

and denote by H_P the completion of H in the norm $\|\cdot\|_P$ induced by $[\cdot, \cdot]_P$. Then $H \subset H_P$ and H is dense in H_P.

Next, it is plain that

$$[PT(t)x, T(t)x] \leqslant [Px, x], \qquad \text{for } x \text{ in } H,$$

where P is positive and $[T(t)]$ is uniformly bounded, is equivalent to

$$2\, Re \cdot [PAx, x] \leqslant 0, \qquad \text{for } x \text{ in } D(A). \tag{2.31}$$

Hence, as in the case of contraction semigroups, we define

$$[x, y]_{P,n} = -[PAx, y] - [x, PAy], \qquad \text{for } x \text{ and } y \text{ in } D(A),$$

from which it follows that

$$[x, x]_{P,n} = -2\, Re \cdot [PAx, x] \geqslant 0, \qquad \text{for } x \text{ in } D(A),$$

by (2.31). Denote $[x, x]_{P,n}$ by $\|x\|^2_{P,n}$ then (2.25) can be rewritten as

$$\|x\|^2_P = \int_0^\infty \|T(t)x\|^2_{P,n}\, dt, \qquad \text{for } x \text{ in } D(A). \tag{2.32}$$

This is an exact analog of (2.17) in which $[T(t)]$ is contractive. Let K_P be the completion of $D(A)$ in the norm $\|\cdot\|_{P,n}$-modulo the null vectors; then $T(t)x$, for x in $D(A)$, belongs to the space $L^2([0, \infty), K_P)$. However, unlike the case of a contraction semigroup, in order to obtain a relationship between strong stability in the norm $\|\cdot\|_P$ and the Left Shift semigroup on the space $L^2([0, \infty), K_P)$, we must extend the semigroup $[T(t)]$, by continuity, from H to H_P. Let $[T_P(t)]$ be the extension; then as before, it can be isometrically represented by the restriction of the Left Shift semigroup to an invariant subspace.

From the above consideration, we can now close the paper with the following results.

THEOREM 8. *Let $[T(t)]$ be a uniformly bounded C_0-semigroup over a Hilbert space H—with inner product $[\cdot, \cdot]$ and norm $\|\cdot\|$, and let A denote its generator. Let $\|\cdot\|_{\text{new}}$ be another norm on H and suppose that it is not equivalent to the original norm.*

Consider the operator equation

$$[PAx, x] + [x, PAx] = -\|x\|^2_{\text{new}}, \qquad \textit{for } x \textit{ in } D(A). \tag{2.33}$$

A necessary and sufficient condition for the existence of a non-negative solution P of (2.33) *is that*

$$\int_0^\infty \|T(t)x\|^2_{\text{new}}\, dt < \infty, \qquad \textit{for } x \textit{ in } H. \tag{2.34}$$

Therefore, a sufficient condition for $[T(t)]$ *to be weakly stable is the existence of a positive solution of* (2.33), *equivalently,*

$$0 < \int_0^\infty \|T(t)x\|^2_{\text{new}}\, dt < \infty, \qquad \textit{for } x \textit{ in } H. \tag{2.35}$$

Proof. Suppose there exists $P \geqslant 0$ satisfying (2.33); then, clearly, for x in the domain of A,

$$[PT(t)x, T(t)x] - [Px, x] = -\int_0^t \|T(\tau)x\|^2_{\text{new}}\, d\tau, \qquad \text{for} \quad t > 0.$$

Therefore (2.34) follows. Conversely, if (2.34) holds then we only must take P to be defined by

$$[Px, x] = \int_0^\infty \|T(t)x\|^2_{\text{new}}\, dt, \qquad x \text{ in } H.$$

Therefore P satisfies (2.33). The last part of the theorem follows easily from Theorem 7. This completes the proof.

It is understood that the new norm is *weaker* than the original norm. Note that if the two norms are equivalent then (2.33) is the familiar Lyapunov Equation. Also, Corollary 2 can be regarded as a special case of Theorem 8 since we can write

$$|[T(t)x, b]|^2 = \|b^*T(t)x\|^2 = [bb^*T(t)x, T(t)x] = \|T(t)x\|^2_{\text{new}}.$$

If the new norm is only defined on the domain of the generator A, as in the case of a contraction semigroup (Theorem 6), then by imposing a further condition we obtain sufficient conditions for strong stability of uniformly bounded semigroups. See also [11] for other extensions of Lyapunov equation.

THEOREM 9. *Let* $[T(t)]$, A, H, *and* $\|\cdot\|$ *be as in Theorem* 8. *Let* $\|\cdot\|_{\text{new}}$ *be a new norm defined on the domain of A and is such that, for some constant* $k > 0$, $k\,\|x\| \leqslant \|x\|_{\text{new}}$ *for all x in* $D(A)$.

A necessary and sufficient condition for the existence of a positive solution P of Eq. (2.33) *is that*

$$\int_0^{\infty} \|T(t)x\|_{\text{new}}^2 \, dt < \infty, \qquad \textit{for } x \textit{ in } D(A). \tag{2.36}$$

Moreover, (2.36) *is sufficient for* $[T(t)]$ *to be strongly stable.*

Proof. The proof of the existence of P is the same as in the proof of Theorem 8, except that here we must extend P by continuity from the dense domain of A to all of H.

Finally, if (2.36) is true then we have

$$k^2 \int_0^{\infty} \|T(t)x\|^2 \, dt \leqslant \int_0^{\infty} \|T(t)x\|_{\text{new}}^2 \, dt < \infty, \qquad \text{for } x \text{ in } D(A).$$

Therefore, by a result of Datko [1],

$$\text{for } x \text{ in } D(A), \qquad \|T(t)x\| \to 0, \qquad t \to \infty.$$

Hence $[T(t)]$ is strongly stable by the fact that the domain of A is dense and the semigroup is uniformly bounded. This finishes the proof.

Finally for further Lyapunov-type results for strong stability we refer to [12].

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